Modified Inverse Iteration Method Using the Side Condition and the Step Length

Case II : Multiple or Close Natural Frequencies

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An efficient numerical method which can analyze the eigenproblem for the large structural system with multiple or close eigenvalues is presented. This method is formulated by applying the accelerated Newton-Raphson method to obtained from the solution of a constrained stationary value problem. The step length used in the accelerated Newton-Raphson method is calculated by the least square concept. This method can calculate the natural frequencies and mode shapes without any numerical instability which may be often encountered in the well-known methods such as the subspace iteration method or the determinant search method which has been widely used for solving eigenvalue problem. The efficiency of this method is verified by comparing convergence and solution time for numerical examples with those of the subspace iteration method and the determinant search method.

Key Words: Multiple or Close Eigenvalues, Finite Element Method, Frequency, Mode Shape, Accelerated Newton-Raphson Technique

1. Introduction

The analysis of structures under dynamic loads is of considerable importance in many fields of engineering. If the dynamic analysis is performed by the mode superposition method, the eigenproblem must be first solved.

In this paper structures with multiple or close eigenvalues are treated. When structures have only distinct eigenvalues, it is explained in the proceeding paper.

If structures with multiple or close natural frequencies, such as multi-span bridges, containment buildings of nuclear power plants, cable-stayed bridges, tires and the structure whose cross-section are symmetric are analyzed by the subspace iteration method (Bathe, 1982 ; Wilson, et. al., 1983) or the determinant search method (Bathe, 1982 ; Bathe, et. al., 1973) which has been mainly used for solving eigenproblems, the numerical instability or the slow convergence may be often encountered.

The objective of this paper is to present an efficient solution method in order to improve numerical stability and increase convergence in case of structures with multiple or close natural frequencies. Numerical examples are presented to show the efficiency of the proposed method.

2. Mehod of Analysis

The following generalized eigenvalue problem is considered in dynamic analysis.

$$K\phi_i=\lambda_i M\phi_i \ (i=1, 2, 3, \cdots, n) \qquad (1)$$

where K and M are the stiffness matrix and the mass matrix of order n, respectively. M is assumed to be positive definite and K positive semidefinite. λ_i is the ith natural frequency squar-

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ed and ϕ_i the corresponding mode shape.

Let a set S consist of s integers, that is, $S=[m, m+1, \cdots, m+s-1]$. s eigenvalues, $\lambda_j(j\in S)$, are equal or close and the corresponding eigenvectors denoted by $\phi_j(j\in S)$. Then the eigenvalue problem with s multiple or close eigenvalues can be written as follows.

$$K\phi_j=\lambda_j M\phi_j \ (j\in S) \tag{2}$$

Let us take s vectors $\psi_j(j\in S)$ which are orthonormal with respect to M and in the neighborhood of the subspace spanned by the eigenvectors $\phi_j(j\in S)$.

With the above definitions, the s dimensional subspace spanned by the eigenvector $\phi_j(j\in S)$ is characterized by the following constrained stationary value problem (Lee, et. al., 1979):
Find the stationary value of

$$w=\sum_{j\in S}\psi_j^T K\psi_j \tag{3}$$

subject to

$$\psi_i^T M\psi_j=\delta_{ij} \ (i, j\in S) \tag{4}$$

where δ_{ij} is the Kronecker delta.

The stationary value problem may be treated by the method of Lagrange multiplers. We have the following Lagrangian

$$L=\sum_{i\in S}\psi_i^T K\psi_i-\sum_{i\in S}\sum_{j\in S}\mu_{ij}(\psi_i^T M\psi_j-\delta_{ij}) \tag{5}$$

where the undetermined multipliers $\mu_{ij}(i, j\in S)$ is symmetric, that is, $\mu_{ij}=\mu_{ji}$.

The problem is solved setting the first partial derivatives of L with respect to the unknowns ψ_j and μ_{ij} equal to zero, i. e.,

$$\frac{\partial L}{\partial \psi_j}=0\ ;\ K\psi_j=\sum_{i\in S}\mu_{ij}M\psi_i \ (j\in S) \tag{6}$$

$$\frac{\partial L}{\partial \mu_{ij}}=0\ ;\ \psi_j^T M\psi_j=\delta_{ij} \ (i, j\in S) \tag{7}$$

We can write Eq. (6) in matrix form as

$$K\psi_j=M\Psi\theta_j \quad (j=m, m+1, \cdots, m+s-1) \tag{8}$$

or collectively

$$K\Psi=M\Psi\Theta \tag{9}$$

In the same way, Eq. (7) can be written as

$$\Psi^T M\Psi=I_s \tag{10}$$

where

$$\Psi=[\Psi_m, \Psi_{m+1}, \cdots, \Psi_{m+s-1}]$$
$$\theta_j^T=\langle \mu_{mj}, \mu_{m+1,j}, \cdots, \mu_{m+s-1,j}\rangle \quad (j=m, m+1, \cdots, m+s-1)$$
$$\Theta=[\theta_m, \theta_{m+1}, \cdots, \theta_{m+s-1}]$$
I_s is the unit matrix of order s.

Let us assume that initial approximate solutions of Eq. (8), $\theta_j^{(0)}$ and $\psi_j^{(0)}(j=m, m+1, \cdots, m+s-1)$, are available. Denote an approximate eigenvalue and the corresponding eigenvector after k iterations by $\theta_j^{(k)}$ and $\psi_j^{(k)}(k=0, 1, 2, \cdots)$. Then, we have

$$r_j^{(k)}=K\psi_j^{(k)}-M\Psi^{(k)}\theta_j^{(k)} \quad (j=m, m+1, \cdots, m+s-1) \tag{11}$$

where the residual vector $r_j^{(k)}$ is not generally zero because of substitution of approximate values into Eq. (8).

In order to make the residual vector a null vector, the Newton-Raphson technique is applied.

$$r_j^{(k+1)}=0=K\psi_j^{(k+1)}-M\Psi^{(k+1)}\theta_j^{(k+1)} \quad (j=m, m+1, \cdots, m+s-1) \tag{12}$$

where

$$\theta_j^{(k+1)}=\theta_j^{(k)}+\Delta\theta_j^{(k)} \quad (j=m, m+1, \cdots, m+s-1) \tag{13}$$
$$\psi_j^{(k+1)}=\psi_j^{(k)}+\Delta\psi_j^{(k)} \quad (j=m, m+1, \cdots, m+s-1) \tag{14}$$

in which $\Delta\theta_j^{(k)}$ and $\Delta\psi_j^{(k)}$ are incremental values for $\theta_j^{(k)}$ and $\psi_j^{(k)}$.

Introducing Eqs. (13) and (14) into Eq. (12) and neglecting the nonlinear terms, $M\Delta\psi_j^{(k)}\Delta\theta_j^{(k)}$, we obtain the linear equations for $\Delta\theta_j^{(k)}$ and $\Delta\psi_j^{(k)}$:

$$-K\psi_j^{(k)}+M\Psi^{(k)}\theta_j^{(k)}=K\Delta\psi_j^{(k)}-M\Delta\Psi^{(k)}\theta_j^{(k)}-M\Psi^{(k)}\Delta\theta_j^{(k)} \quad (j=m, m+1, \cdots, m+s-1) \tag{15}$$

If the $\lambda_j(j\in S)$ are multiple or close eigenvalues, the off-diagonal elements of Θ are zero or very small compared with its diagonal ones (Lee, et. al, 1979), thus the 2nd term in right hand side of Eq. (15) may be approximated by $\mu_{jj}^{(k)}M\Delta\psi_j^{(k)}$, yielding

$$-K\psi_j^{(k)}+M\Psi^{(k)}\theta_j^{(k)}=K\Delta\psi_j^{(k)}-\mu_{jj}^{(k)}M\psi_j^{(k)}-M\Psi^{(k)}\Delta\theta_j^{(k)}$$

$$(j=m,\ m+1,\ \cdots,\ m+s-1) \tag{16}$$

Because there are only n equations with $n+s$ unknowns, s components of $\Delta\theta_j^{(k)}$ and n components of $\Delta\psi_j^{(k)}$, a side condition must be introduced for the solution of Eq. (16). The side condition to arrive at a set of $n+s$ equations with $n+s$ unknowns is

$$(\Psi^{(k)})^T M\Delta\psi_j^{(k)}=0 \quad (j=m,\ m+1,\ \cdots,\ m+s-1) \tag{17}$$

This means that the allowable changes in the approximate eigenvector are orthogonal to the latest subspace spanned by the approximate eigenvectors with respect to the mass matrix. This prevents unlimited drift in the eigenvector which is, after all, not determined in magnitude.

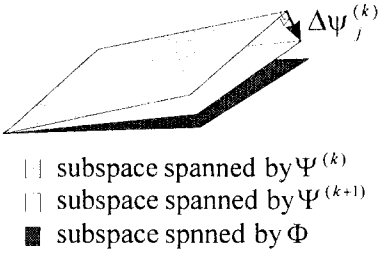


Fig. 1 The side condition(if M=I)

Writing Eqs. (16) and (17) in matrix form, we get

$$\begin{bmatrix} K-\mu_{jj}^{(k)}M & -M\Psi^{(k)} \\ -(\Psi^{(k)})^T M & 0 \end{bmatrix}\begin{Bmatrix} \Delta\psi_j^{(k)} \\ \Delta\theta_j^{(k)} \end{Bmatrix} = -\begin{Bmatrix} r_j^{(k)} \\ 0 \end{Bmatrix} \quad (j=m,\ m+1,\ \cdots,\ m+s-1) \tag{18}$$

$$\theta_j^{(k+1)}=\theta_j^{(k)}+\Delta\theta_j^{(k)} \quad (j=m,\ m+1,\ \cdots,\ m+s-1) \tag{13}$$

$$\psi_j^{(k+1)}=\psi_j^{(k)}+\Delta\psi_j^{(k)} \quad (j=m,\ m+1,\ \cdots,\ m+s-1) \tag{14}$$

The above equations are the numerical method derived by the Newton-Raphson technique.

The convergence of this method can be improved by the accelerated Newton-Raphson technique as in the proceeding paper(Case I : Distinct Natural frequencies).

$$\begin{bmatrix} K-\mu_{jj}^{(0)}M & -M\Psi^{(k)} \\ -(\Psi^{(k)})^T M & 0 \end{bmatrix}\begin{Bmatrix} \Delta\psi_j^{(k)} \\ \Delta\theta_j^{(k)} \end{Bmatrix} = -\begin{Bmatrix} r_j^{(k)} \\ 0 \end{Bmatrix} \quad (j=m,\ m+1,\ \cdots,\ m+s-1) \tag{19}$$

$$\theta_j^{(k-1)}=\theta_j^{(k)}+\Delta\theta_j^{(k)} \quad (j=m,\ m+1,\ \cdots,\ m+s-1) \tag{13}$$

$$\psi_j^{(k+1)}=\psi_j^{(k)}+\alpha_j^{(k)}\Delta\psi_j^{(k)} \quad (j=m,\ m+1,\ \cdots,\ m+s-1) \tag{20}$$

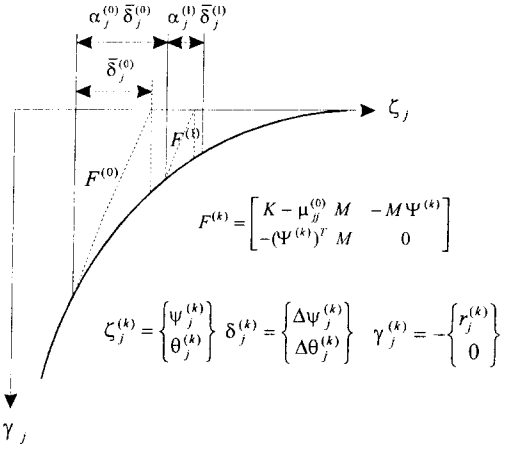


Fig. 2 The accelerated Newton-Raphson method

$\alpha_j^{(k)}$ in Eq. (20) is a value to minimize the norm of the residual vector, $r_j^{(k+1)}$, and can be evaluated by using the least square method as follows.

$$\frac{\partial}{\partial\alpha_j^{(k)}}\{(r_j^{(k+1)})^T r_j^{(k+1)}\}=0 \quad (j=m,\ m+1,\ \cdots,\ m+s-1) \tag{21}$$

$$\alpha_j^{(k)}=-\frac{(\Delta\psi_j^{(k)})^T(K-\mu_{jj}^{(k+1)}M)(K-\mu_{jj}^{(k+1)}M)\psi_j^{(k)}-(\Delta\psi_j^{(k)})^T(K-\mu_{jj}^{(k+1)}M)\left(\sum_{\substack{i=m\\ i\neq j}}^{m+s-1}\mu_{ij}^{(k+1)}M\psi_i^{(k)}\right)}{(\Delta\psi_j^{(k)})^T(K-\mu_{jj}^{(k+1)}M)(K-\mu_{jj}^{(k+1)}M)\Delta\psi_j^{(k)}} \tag{22}$$

Note that $\mu_{ij}^{(k+1)}(i,\ j\in S)$ and $\Delta\psi_j^{(k)}$ have been obtained by Eq. (13) and Eq. (19) respectively.

If the order of the system is n, and the bandwidths of the stiffness matrix and mass matrix are m_a and m_b respectively, the number of operations for evaluation $\alpha_j^{(k)}$ in the first iteration step

is $2nm_a+2nm_b+(s+5)n+1$. This is large compared to $(s+4)nm_a+2snm_b+(1/2)n(s^2+7s+4)$ which is required in each iteration step in Eq. (19). However, only the number of $7n+1$ operations is required to evaluate $\alpha_j^{(k)}$ after the 2nd iteration, which is negligible, because we use computational results in the previous iteration. Thus, solution time of the proposed method is decreased by improving convergence.

As k increases, the incremental values $\Delta\theta_j^{(k)}$ and $\Delta\psi_j^{(k)}$ will vanish. Then from Eq. (19)

$$\lim_{k\to\infty} r_j^{(k)}=\lim_{k\to\infty}(K\psi_j^{(k)}-M\Psi^{(k)}\theta_j^{(k)})=0 \quad (j=m,\ m+1,\ \cdots,\ m+s-1) \tag{23}$$

Letting

$$\theta_j=\lim_{k\to\infty}\theta_j^{(k)} \quad (j=m,\ m+1,\ \cdots,\ m+s-1)$$
$$\psi_j=\lim_{k\to\infty}\psi_j^{(k)} \quad (j=m,\ m+1,\ \cdots,\ m+s-1) \tag{24}$$

Eq. (23) can be written as

$$K\Psi=M\Psi\Theta \tag{25}$$

where $\Psi=[\psi_m,\ \psi_{m+1},\ \cdots,\ \psi_{m+s-1}]$, $\Theta=[\theta_m,\ \theta_{m+1},\ \cdots,\ \theta_{m+s-1}]$

If the eigenvalues $\lambda_j(j\in S)$ are multiple, the values of the off-diagonal elements of Θ are all zero and its diagonal elements have an equal value which is the desired multiple eigenvalue.

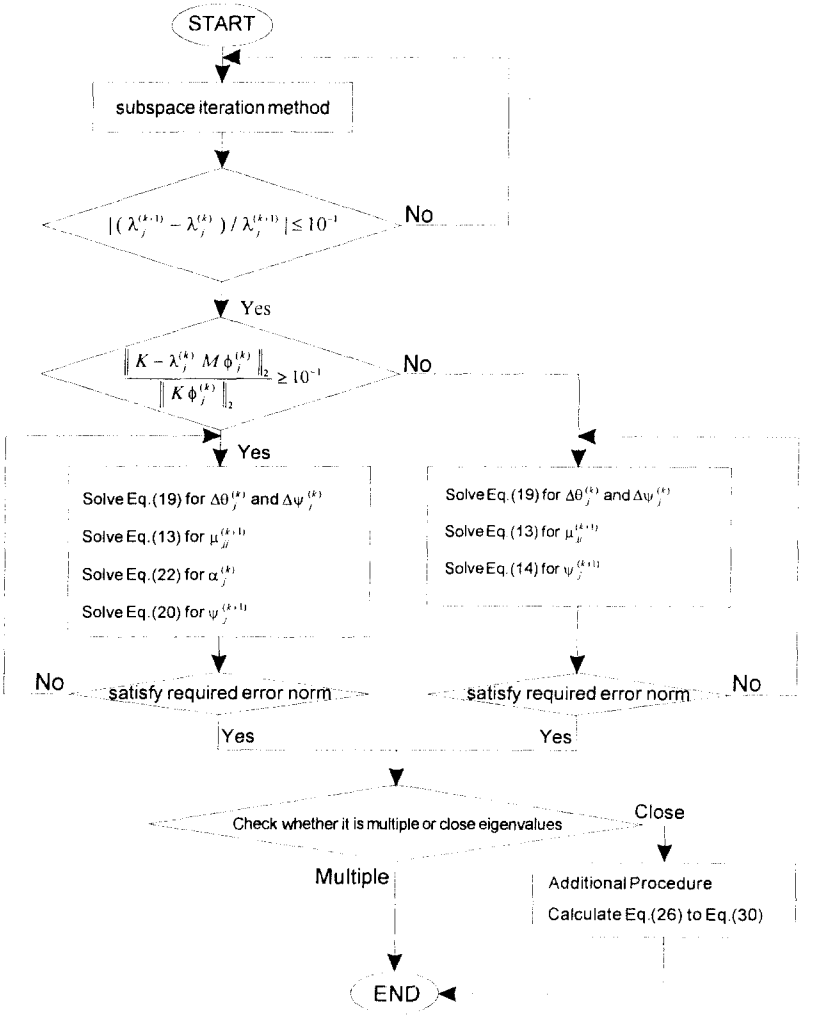


Fig. 3 Algorithm for the proposed method

Moreover, the vectors in Ψ are the corresponding eigenvectors.

If the eigenvalues are merely close, additional operations are required because the values of the off-diagonal elements of Θ are not zero. These additional operations can be written as follows. To find the true eigenvectors the vectors in Ψ should be rotated in the subspace of Ψ. A rotation matrix is found by solving a small eigenvalue problem. The derivation of the small eigenvalue problem is as follows. We can write Eq. (2) collectively as

$$K\Phi = M\Phi\Lambda \tag{26}$$

where $\Phi = [\phi_m, \phi_{m+1}, \cdots, \phi_{m+s-1}]$, $\Lambda = diag(\lambda_m, \lambda_{m+1}, \cdots, \lambda_{m+s-1})$. Now let

$$\Phi = \Psi Z \tag{27}$$

where Z is the unknown rotation matrix of order s. Introducing Eq. (27) into Eq. (26), we get

$$K\Psi Z = M\Psi Z\Lambda \tag{28}$$

Postmultiplying Eq. (25) by the matrix Z yields

$$K\Psi Z = M\Psi\Theta Z \tag{29}$$

Premultiplying Eqs. (28) and (29) by the transpose of the matrix Ψ and using $\Psi^T M\Psi = I_s$ of Eq. (10), we obtain the special eigenvalue problem of order s

$$\Theta Z = Z\Lambda \tag{30}$$

where Θ is symmetric and of order s, the number of close eigenvalues, which is usually small.

The eigenvalue problem, Eq. (30) can be easily solved by any suitable technique such as the Jacobi method, yielding the desired eigenvalues in Λ and the matrix Z, which in turn gives the eigenvectors Θ by Eq. (27). The number of operations required for the solution of Eq. (30) is very small, since s is small.

Some of the eigenvalues and corresponding eigenvectors of interest may be missed when the initial approximations are not suitable. In order to check whether this occurs, the Sturm-sequence property (Bathe, 1982) may be applied. A computed eigenvalue can be checked using the above property with negligible extra computation, since the decomposition of the matrix $(K - \mu_{jj}^{(0)} M)$ has already been carried out during the procedure for the solution of Eq. (19). If some of the eigenvalues of interest are detected to be missing, the solutions can be found by the proposed method.

3. Numerical Examples

The simply-supported plane and the cooling tower are analyzed to verify the efficiency of the proposed method. By using three methods separately, the subspace iteration method, the determinant search method and the proposed method, each convergence and solution time (CPU time) used to calculate 10 eigenpairs with error norm of 1.E-09 are compared, where the error norm (Bathe, 1982) is computed by

$$error\ \ norm = \frac{\| (K - \lambda_j^{(k)} M)\phi_j^{(k)} \|_2}{\| K\phi_j^{(k)} \|_2} \tag{31}$$

Especially to get the best results we generated the 20 starting iteration vectors using the Lanczos method and applied the accelerated scheme (Bathe, et. al., 1980) to the subspace iteration method. Intermediate results with relative error of 1.E-01 in the subspace iteration method are used as initial values of the proposed method. The relative error (Bathe, 1982) in the subspace iteration method is computed as follows

$$relative\ \ error = \left| \frac{\lambda_j^{(k+1)} - \lambda_j^{(k)}}{\lambda_j^{(k+1)}} \right| \tag{32}$$

$\alpha_j^{(k)}$ is applied to the eigenpair whose error norm is over 1.E-01. All runs are executed in the IRIS4D-20-S17 with 10 Mips and 0.9 Mfolps.

3.1 Simply-supported plane structure

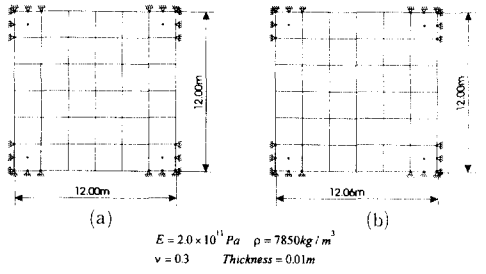
The simply-supported plane structures shown Fig. 4 consist of 36 nine-node shell elements, 169 nodes and 701 degrees-of-freedom. The stiffness matrix and mass matrix have the mean half-bandwidths of 89.

3.1.1 Multiple natural frequencies

The simply-supported plane structure with multiple natural frequencies is shown in Fig. 4(a). $\alpha_j^{(k)}$ is applied to the 6th, the 8th and the 10th eigenpair with error norm exceeding 1.E-01.

Each solution time for three methods to have 10

eigenpairs with the error norm of 1.E-09 is summarized in Table 1. If we let the solution time for the proposed method be 1, it takes 1.5 times for the accelerated subspace iteration method, 6.5 times for the determinant search method. For each solution method the convergence of the 8th eigenpair to which $\alpha_j^{(k)}$ is applied is represented in Fig. 5. As shown in the above figure, we can see that the convergence of the proposed method is much superior to that of the accelerated subspace iteration and of the determinant search method.



(a) Multiple natural frequencies
(b) Close natural frequencies

Fig. 4 The simply-supported plane

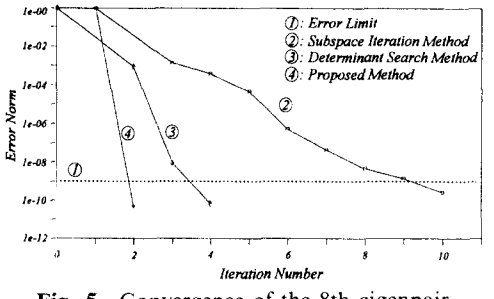


Fig. 5 Convergence of the 8th eigenpair

Table 1 Solution time (CPU time, sec) of simply supported plane(multiple case)

Methods	Solution time (ratio)
Proposed method	172.4(1.0)
Acc. Supspace iteration method	274.6(1.6)
Determinant search method	1118.6(6.5)

Table 2 Solution time (CPU time, sec) of simply supported plane(close case)

Methods	Solution time (ratio)
Proposed method	177.4(1.0)
Acc. Supspace iteration method	291.6(1.7)
Determinant search method	832.1(4.7)

3.1.2 Close natural frequencies

The simply-supported plane structure with close natural frequencies is shown Fig. 4(b). $\alpha_j^{(k)}$ is applied to the 7th, the 8th and the 10th eigenpair with error norm exceeding 1.E-01.

Each solution time for three methods to have 10 eigenpairs with the error norm of 1.E-09 is summarized in Table 2. If we let the solution time for the proposed method be 1, it takes 1.7 times for the accelerated subspace iteration method, 4.7 times for the determinant search method. For each solution method the convergence of eigenpairs to which $\alpha_j^{(k)}$ is applied is represented from Fig. 6 to Fig. 7. As shown in the above figures, we can see that the convergence of the proposed method is superior to that of the acceler-

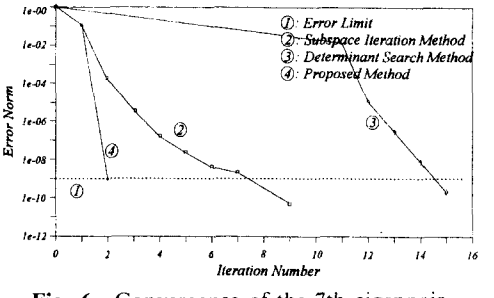


Fig. 6 Convergence of the 7th eigenpair

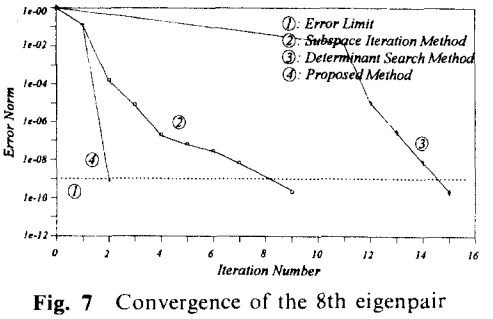


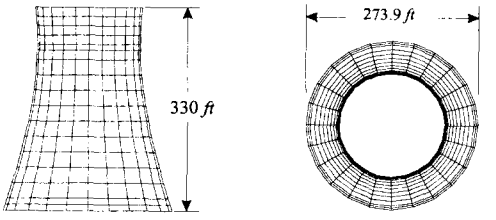
Fig. 7 Convergence of the 8th eigenpair

ated subspace iteration and of the determinant search method.

3.2 Cooling tower structure

The cooling tower structure shown Fig. 8 consists of 408 four-node shell elements, 432 nodes and 2448 degrees-of-freedom. The stiffness and the mass matrix have the mean half-bandwidths of 201. $\alpha_j^{(k)}$ is applied to the 8th and the 10th eigenpair with error norm exceeding 1.E-01.

Each solution time for two solution methods to have 10 eigenpairs with the error norm of 1.E-09 is summarized in Table 3. Determinant search



(a) Plan (b) Elevation

Fig. 8 Cooling tower

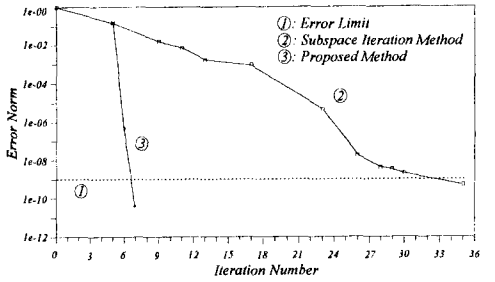


Fig. 9 Convergence of the 8th eigenpair

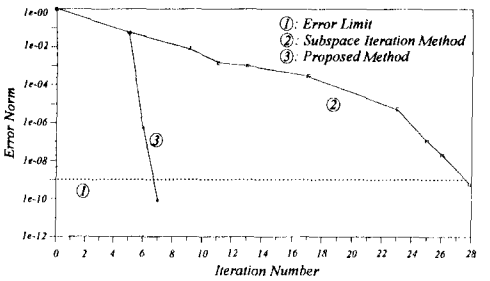


Fig. 10 Convergence of the 10th eigenpair

Table 3 Solution time (CPU time, sec) of cooling tower(multiple case)

Methods	Solution time (ratio)
Proposed method	3067.7(1.0)
Acc. Supspace iteration method	6182.5(2.0)

method is not applied because it did not give us the good results. If we let the solution time for the proposed method be 1, it takes 2.0 times for the accelerated subspace iteration method. For each solution method the convergence of eigenpairs to which $\alpha_j^{(k)}$ is applied is represented from Fig. 9 to Fig. 10.

4. Conclusions

This paper proposes an efficient numerical method for the multiple or close eigenvalue problems using the accelerated Newton-Raphson method. As shown in the examples, the proposed method has the characteristics as follows.

(1) The proposed method is a general technique which can identify the eigenpairs of a structure efficiently without any numerical instability in case of multiple or close eigenvalues as well as distinct eigenvalues.

(2) The proposed method will not be affected by the eigenpairs previously calculated, because each eigenpair is essentially obtained independently.

(3) The decrement of convergence has been found in some cases using the accelerated Newton-Raphson method because it uses a constant step length without considering the shape of characteristic function. With this proposed method, the disadvantage stated above can be overcomed by taking the adequate step length($\alpha_j^{(k)}$) which includes the shape of the characteristic function to minimize the norm of the residual vector in each iteration step.

In brief, the proposed method is an efficient technique which can obtain the eigenpairs of a structure especially involving multiple or close eigenvalues with numerical stability.

5. Acknowledgement

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6. References

Bathe, K. J., 1982, *Finite Element Procedures in Engineering Analysis*, Prentice-Hall.

Bathe, K. J. and Ramaswamy, S., 1980, "An Accelerated Subspace Iteration Method," *Computer Methods in Appl. Mech. and Eng.*, Vol. 23, pp. 313~331.

Bathe, K. J. and Wilson, E. L., 1973, "Eigensolution of Large Structural Systems with Small Bandwidth," *J. Eng. Mech. Div.*, Vol. 99, pp. 467~479.

Habibulah, A. and Wilson, E. L., 1989, *SAP90TM Sample Example and Verification Manual*, Computers and Structures, Inc.

Lee, I. W., Jung, H. J. and Kim, M. C., 1994, "An Efficient Free Vibration Analysis of Structures with Multiple Natural Frequencies," *Proceedings of the Korea Society of Civil Engineers*, pp. 135~138.

Lee, I. W. and Robinson, A. R., 1979, "Solution techniques for Large Eigenvalue Problems in Structural Dynamics," *Structural Research Series No.* 462, University of Illinois.

Wilson, E. L. and Itoh, T., 1983, "An Eigensolution strategy for Large Systems," *Comp. Struct.*, Vol. 16, pp. 259~265.

1983, *ADINA System Verification Manual*, ADINA Engineering, Inc.